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Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes

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Abstract

We consider the geometric transition and compute the all-genus topological string amplitudes expressed in terms of Hopf link invariants and topological vertices of Chern-Simons gauge theory. We introduce an operator technique of 2-dimensional CFT which greatly simplifies the computations. We in particular show that in the case of local Calabi-Yau manifolds described by toric geometry basic amplitudes are written as vacuum expectation values of a product vertex operators and thus appear quite similar to the Veneziano amplitudes of the old dual resonance models. Topological string amplitudes can be easily evaluated using vertex operator algebra.

1 Introduction

Recently the method of large N duality and the geometric transition has been applied to deriving all-genus amplitudes for topological string theory compactified on certain local Calabi-Yau manifolds [1, 2, 3, 4, 5]. Here one starts from the deformed conifold side and makes use of the Chern-Simons gauge theory and its link invariants and computes the topological string amplitudes of the resolved conifold side. When one makes a suitable choice of the non-compact Calabi-Yau manifolds, e.g. local $\mathbf{F}_0$ or its generalizations, one obtains results which reproduces the well-known solutions of the 4-dimensional $\mathcal{N} = 2$ SUSY gauge theories by Seiberg and Witten [6].

In a previous article [7] we have evaluated rigorously the topological string amplitudes by proving the propositions proposed by [8, 9, 10] and have shown that the results agree exactly with the formula of Nekrasov [11] when his parameters $\beta, \hbar$ are identified with the Kähler parameters of local Calabi-Yau manifolds. Thus Nekrasov's formula not only reproduces the Seiberg-Witten solution in the 4-dimensional limit $\beta, \hbar \to 0$ but encodes the entire information on the number of holomorphic curves of arbitrary degrees and genera in local Calabi-Yau manifolds when $\beta, \hbar \neq 0$. (For related more recent works see [12, 13, 14]).

In these computations the Hopf link invariants of Chern-Simons theory are expressed in terms of Schur functions associated with the representations of Wilson lines and we have to evaluate the sum of the product of link invariants over all representations or Young tableaus. We have used certain identities of (skew) Schur functions in carrying out the computations. It is well-known, however, Schur functions have a simple representation in terms of the vertex operators and there is also a one to one correspondence between Young tableaus with the elements of the free fermion Fock space. Thus by making use of the operator method of 2-dimensional CFT we can greatly streamline the computations of Chern-Simons theory.

In this article we would like to show that in fact the computation of topological string amplitude using Chern-Simons theory is very much simplified and reduced to evaluating

Veneziano-like amplitudes of the old dual models when the operator method is used; the basic amplitude of the theory becomes the vacuum expectation value of a product of vertex operators and this observation greatly facilitates the calculations.

Such an operator technique has also been discussed recently in detail in the context of the B-model version of the geometric transition [15]. Here our presentation corresponds to the A-model side of the story.

2 Free fields and the vertex operator

Let us first introduce a free boson in two dimensions, or the infinite dimensional Heisenberg algebra with the generators α_n which satisfy the commutation relation:

$$[\alpha_m, \alpha_n] = m\delta_{m,-n} \ . \tag{2.1}$$

Our convention is that $\alpha_n(n > 0)$ are the annihilation operators and $\alpha_{-n}(n > 0)$ are the creation operators. (In the following the zero mode α_0 plays no role.) We define the annihilation Γ_+ and the creation Γ_- part of the vertex operator as follows;

$$\Gamma_\pm(t_n) := \exp\left(\sum_n t_n \alpha_{\pm n}\right) \ . \tag{2.2}$$

They satisfy

$$\Gamma^*_\pm(t_n) = \Gamma_\mp(t_n) \ , \qquad \Gamma_+(t_n)|0\rangle = |0\rangle \ , \tag{2.3}$$

and the commutation relation

$$\Gamma_+(t_n)\Gamma_-(s_n) = \exp(\sum_n n t_n s_n)\Gamma_-(s_n)\Gamma_+(t_n) \ . \tag{2.4}$$

We specify the values for the parameters $\{t_n\}, \{s_n\}$ soon below.

It is well-known that the free fermion description of a Young diagram of a representation R is given by an element in the free fermion Fock space which is obtained by filling all the energy levels at

$$-(\mu_i^R - i + \frac{1}{2}), \quad i = 1, 2, 3, \cdots \tag{2.5}$$

where μ_i^R is the length of the i-th row of the Young tableau of representation R.

Let us denote by $\psi_n^*\,(\psi_m)$ the fermion creation (annihilation) operators at the level $n\,(m)$. Boson-fermion correspondence is given by the usual formula

$$i\partial_z\phi(z) =: \psi(z)^*\psi(z) :, \quad \phi(z) = i\sum_n \frac{1}{n}\alpha_n z^{-n}, \tag{2.6}$$

$$\psi(z)^* = \sum_n \psi_n^* z^{-n-1/2}, \quad \psi(z) = \sum_n \psi_n z^{-n-1/2}. \tag{2.7}$$

The Dirac sea corresponding to the trivial representation $\bullet$ is then given by

$$|\text{sea}\rangle \equiv |0\rangle = \psi_{1/2}^*\psi_{3/2}^*\psi_{5/2}^*\psi_{7/2}^*\psi_{9/2}^*\cdots|0\rangle\rangle\ . \tag{2.8}$$

If one considers a non-trivial representation, for instance, an anti-symmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, the corresponding state is given by

$$|v_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\rangle = \psi_{-1/2}^*\psi_{1/2}^*\psi_{5/2}^*\psi_{7/2}^*\psi_{9/2}^*\cdots|0\rangle\rangle\ . \tag{2.9}$$

If instead one considers the symmetric representation $\square\square$, one obtains

$$|v_{\square\square}\rangle = \psi_{-3/2}^*\psi_{3/2}^*\psi_{5/2}^*\psi_{7/2}^*\psi_{9/2}^*\cdots|0\rangle\rangle\ . \tag{2.10}$$

Now let us set the values of the parameters $\{t_n\}$ of the vertex operator as the power sum of the basic variables $\{x_i\}$

$$t_n = \frac{1}{n}p_n = \frac{1}{n}\sum_{i=1} x_i^n \tag{2.11}$$

and introduce the notation

$$V_\pm(x_i) := \Gamma_\pm(t_n = \frac{1}{n}p_n(x_i))\ . \tag{2.12}$$

Then one defines the Schur functions by the matrix elements

$$s_R(x_i) = \langle v_R|V_-(x_i)|0\rangle. \tag{2.13}$$

In the case of the anti-symmetric representation of second rank (2.9), one finds

$$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_i) = \sum_{i<j} x_i x_j. \tag{2.14}$$

Similarly, in the case of the symmetric representation of second rank (2.10), one has

$$s_{\square\square}(x_i) = \sum_i x_i^2 + \sum_{i<j} x_i x_j. \tag{2.15}$$

Thus we obtain elementary (complete) symmetric polynomials of the second order which in fact agree with the Schur functions associated with these representations. The formula (2.13) gives the general prescription of representing the Schur functions in terms of vertex operators.

It is easy to generalize (2.13) to the case of the skew Schur functions and represent them as the matrix elements of the vertex operator

$$s_{R/Q}(x) = \langle R|V_-(x_i)|Q\rangle = \langle Q|V_+(x_i)|R\rangle \ . \tag{2.16}$$

Here the Young tableau of the representation Q must be contained in that of R.

From the basic relation (2.16) it is easy to see that the identity for the summation of the product of the skew Schur functions (A.14), for instance, can be reproduced using the commutation relation of the vertex operators. For example,

$$\begin{aligned}
\sum_R s_{R/Q}(x) s_{R/T}(y) &= \sum_R \langle Q|V_+(x_i)|R\rangle\langle T|V_+(y_i)|R\rangle \\
&= \langle Q|V_+(x_i)V_-(y_i)|T\rangle \\
&= \prod_{i,j\geq 1}(1-x_i y_j)^{-1}\langle Q|V_-(y_i)V_+(x_i)|T\rangle \\
&= \prod_{i,j\geq 1}(1-x_i y_j)^{-1}\sum_U \langle Q|V_-(y_i)|U\rangle\langle U|V_+(x_i)|T\rangle \\
&= \prod_{i,j\geq 1}(1-x_i y_j)^{-1}\sum_U s_{Q/U}(y) s_{T/U}(x) \ .
\end{aligned} \tag{2.17}$$

Here we have used

$$\prod_{i,j\geq 1}(1-x_i y_j)^{-1} = \exp\left[\sum_{n=1} \frac{1}{n} p_n(x_i) p_n(y_j)\right] \ . \tag{2.18}$$

In the computation of topological string amplitudes, we will need the "energy" operator L_0 appearing in the propagator. In terms of the oscillator of the free boson L_0 is given

by

$$L_0 := \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \ . \tag{2.19}$$

Note that L_0 has no zero mode. The commutation relation

$$[L_0, \alpha_n] = -n\alpha_n \ , \tag{2.20}$$

implies that

$$Q^{L_0}\alpha_n Q^{-L_0} = Q^{-n}\alpha_n \ . \tag{2.21}$$

Therefore we obtain

$$Q^{L_0}V_{\pm}(x_i) = V_{\pm}(Q^{\mp 1}x_i)Q^{L_0} \ . \tag{2.22}$$

3 Sample calculation of Veneziano-like amplitudes

The $SU(N)$ supersymmetric gauge theory with eight super charges is geometrically engineered by local Calabi-Yau manifolds with ALE space of A_{N-1} type fibered over $\mathbf{P}^1$ [16, 17, 18, 19]. The fiber consists of a chain of $(N-1)$ rational curves which give a minimal resolution of A_{N-1} singularity in (complex) two dimensions. The dual toric diagram of this local Calabi-Yau geometry is given by the ladder diagram with N horizontal legs with 4 external lines as depicted in Fig.1. By cutting in the middle across N legs as in [10], we obtain a pair of tree diagrams with $N+2$ legs each. These graphs remind us of the multi-peripheral diagrams of the old dual resonance model. In fact as we will see shortly the computation of topological string amplitude using the Hopf link invariant and topological vertex becomes completely analogous to the computation of the Veneziano amplitudes, once we use the operator formalism in terms of free fields.

We first recall that the Hopf link invariant $W_{R_1R_2}$ represents the expectation value of a pair of linked Wilson lines with representations R_1 and R_2 in Chern-Simons gauge theory and is given by the product of Schur functions with the variables $\{x_i\}$ specialized at particular values [20, 7]

$$W_{R_1R_2}(q) = s_{R_1}(q^{\rho})s_{R_2}(q^{{\mu_i}^{R_1}+\rho}). \tag{3.1}$$

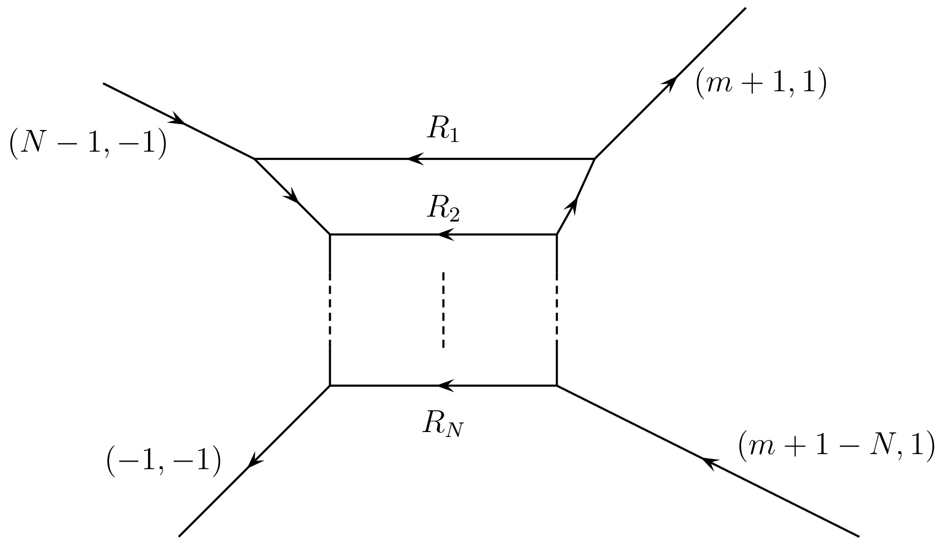

Figure 1: Ladder diagram for $SU(N)$ gauge theory. Note that there are $N+1$ possible toric diagrams ($m = 0, \cdots N$). m is related to the coefficients of the Chern-Simons coupling in five dimensions.

Here $q^{\mu+\rho}$ and q^{ρ} means that we specialize the values of $\{x_i\}$ at

$$x_i = q^{\mu_i - i + \frac{1}{2}} \quad \text{and} \quad x_i = q^{-i+\frac{1}{2}}, \quad i = 1, 2, 3, \cdots \tag{3.2}$$

respectively. On the other hand the topological vertex [5] is defined by

$$C_{R_1,R_2,R_3} = q^{\frac{\kappa_{R_2}}{2} + \frac{\kappa_{R_3}}{2}} s_{R_2^t}(q^{\rho}) \sum_{Q_3} s_{R_1/Q_3}(q^{\mu^{R_2^t}+\rho}) s_{R_3^t/Q_3}(q^{\mu^{R_2}+\rho}) \ . \tag{3.3}$$

R^t denotes the conjugate representation of R. κ_R and ℓ_R of the representation R are defined as usual

$$\kappa_R = \ell_R + \sum_{j=1}^{d(R)} \mu_j(\mu_j - 2j) \ , \quad \ell_R = \sum_{j=1}^{d(R)} \mu_j \ , \tag{3.4}$$

where $d(R)$ is the depth of the representation R. Details of the Hopf invariant and topological vertex are relegated to the Appendix.

Let us now compute the amplitudes of the "half" of the $SU(N)$ ladder diagram. See Fig.2. The value of this amplitude is independent of the parameter m of Fig.1. We assign

to each of the cut N legs associated with the base $\mathbf{P}^1$ the representations $R_1, R_2, \cdots R_N$ which are incoming to the vertex by convention.

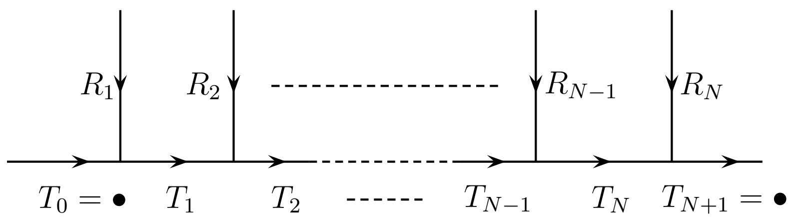

Figure 2: Veneziano-like diagram from cutting in half the $SU(N)$ ladder diagram. Note that we respect topology only and ignore the slopes of the edges.

To the edges corresponding to the components of the fiber we assign representations $T_0, T_1, \cdots T_{N-1}, T_N$ with T_0 and T_N being the trivial representations. Then the amplitude is expressed as the product of N topological vertices of the form

$$C_{T_{k-1}R_kT_k^t} \cdot (-1)^{\ell_{T_k}} = (-1)^{\ell_{T_k}} q^{-\kappa_{T_k}/2} s_{R_k}(q^\rho) \sum_{U_k} s_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) s_{T_k/U_k}(q^{\mu_{R_k}+\rho}) \ , \quad (3.5)$$

together with the propagators $Q_k^{\ell_{T_k}} := e^{-t_k \cdot \ell_{T_k}}$. Note that we have an extra sign factor $(-1)^{\ell_{T_k}}$ since T_k is outgoing in our convention. The parameters $t_k (k = 1 \cdots N-1)$ are the Kähler moduli of the k-th $\mathbf{P}^1$ in the chain of rational curves and related to the $SU(N)$ gauge theory parameters a_k by

$$Q_k = \exp(-2R(a_k - a_{k+1})) \ . \quad (3.6)$$

It is important to take into account the framing factor $(-1)^{n\ell_T} q^{-n\kappa_T/2}$, $n \in \mathbf{Z}$ for each $\mathbf{P}^1$ of the fiber. Since the A_{N-1} singularity is fibered over the base, each $\mathbf{P}^1$ of the fiber has a zero self-intersection number and the framing index n becomes $n = -1$. Thus the factor coming from the framing cancels the factor $(-1)^{\ell_{T_k}} q^{-\kappa_{T_k}/2}$ of (3.5) and we obtain the amplitude

$$K_{\{R_i\}}^{SU(N)} = \sum_{T_1, \cdots, T_{N-1}} \prod_{k=1}^{N} Q_k^{\ell_{T_K}} s_{R_k}(q^\rho) \sum_{U_k} s_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) s_{T_k/U_k}(q^{\mu_{R_k}+\rho}) \ . \quad (3.7)$$

We recall that $q^{\mu+\rho}$ stands for the substitution $x_i = q^{\mu_i - i + 1/2}$. Accordingly let us introduce a notation

$$V_\pm^{[R]}(q) := V_\pm(x_i = q^{\mu_i^R - i + 1/2}) \ . \tag{3.8}$$

We then have

$$\begin{aligned} & Q_k^{\ell_{T_k}} \sum_{U_k} s_{T_{k-1}/U_k}(q^{\mu_{R_k^t}+\rho}) s_{T_k/U_k}(q^{\mu_{R_k}+\rho}) \\ = & Q_k^{\ell_{T_k}} \sum_{U_k} \langle T_{k-1}| V_-^{[R_k^t]}(q) |U_k\rangle \langle U_k | V_+^{[R_k]}(q) | T_k \rangle \\ = & \langle T_{k-1}| V_-^{[R_k^t]}(q) V_+^{[R_k]}(q) Q_k^{L_0} | T_k \rangle \ . \end{aligned} \tag{3.9}$$

Substituting this, we obtain the following Veneziano-like amplitude for topological string;

$$\begin{aligned} K_{\{R_i\}}^{SU(N)} &= \sum_{T_1,\cdots,T_{N-1}} \prod_{k=1}^N s_{R_k}(q^\rho) \langle T_{k-1} | V_-^{[R_k^t]} V_+^{[R_k]} Q_k^{L_0} | T_k \rangle \\ &= \prod_{k=1}^N s_{R_k}(q^\rho) \cdot \langle 0 | \prod_{k=1}^N V_-^{[R_k^t]} V_+^{[R_k]} Q_k^{L_0} | 0 \rangle \ , \\ &= \prod_{i=1}^N \dim_q R_i \cdot \langle 0 | \prod_{k=1}^N V_-^{[R_k^t]} V_+^{[R_k]} Q_k^{L_0} | 0 \rangle \ . \end{aligned} \tag{3.10}$$

We have used the fact that T_0 and T_{N+1} are the trivial representations and $s_{R_k}(q^\rho) = \dim_q R_k$. (3.10) is the main result of this paper.

Using the commutation relations (2.4) and (2.22) we obtain

$$\begin{aligned} K_{\{R_i\}}^{SU(N)}(Q_i) &= \prod_{i=1}^N \dim_q R_i \cdot \prod_{1\le m<\ell\le N} \prod_{1 \le i,j} \left(1 - (\prod_{n=m}^{\ell-1} Q_n) \cdot q^{h_{R_m R_\ell^t}(i,j)} \right)^{-1} \ , \\ = \prod_{i=1}^N \dim_q R_i \cdot & \prod_{1\le m<\ell\le N} \prod_{k}^{\infty} \left(1 - (\prod_{n=m}^{\ell-1} Q_n) \cdot q^k \right)^{-k} \prod_{k} \left(1 - (\prod_{n=m}^{\ell-1} Q_n) \cdot q^k \right)^{-C_k(R_m, R_\ell^t)} . \end{aligned} \tag{3.11}$$

$h_{R_1R_2}(i,j)$ denotes the relative hook length (A.17) and in deriving the 2nd line we have used the lemma in Appendix.

This is a formula conjectured in [10] and proved recently in [12] using Schur function identities. Topological string amplitudes can be immediately constructed from the above

formula; taking the square of $K^{SU(N)}_{\{R_i\}}$ and multiply the propagators $e^{-t_B \sum_i l_{R_i}}$ and sum over all representations $\{R_i\}$. One then finds the exact agreement with the formula of Nekrasov for $SU(N)$ gauge theory.

4 Discussions

In this paper we have introduced operator techniques in evaluating Chern-Simons amplitudes in the theory of geometric transition. We find that a free field representation of Schur functions and Young tableaus greatly simplifies the calculation. In the end the computation boils down to evaluating Veneziano type amplitudes of old dual resonance models. In fact the formula (3.10) is exactly the form of a vacuum value of a product of vertex operators $V_-^{[R^t]} V_+^{[R]}$: vertex operator carries the quantum number of a representations R via the specialization of its variables $\{x_i = q^{\mu_i^R - i + 1/2}\}$. The amplitudes can then be easily evaluated by using the vertex operator algebra.

It seems that this is a somewhat simpler structure than suggested by [5] based on the consideration of the B-model version of the geometric transition. Actually it is rather mysterious why we end up with exactly the free field theory structure in dealing with computation of the all-genus topological string amplitudes. A possible answer is that the free field nature of the problem originates from the fact that we are dealing with the local Calabi-Yau manifolds where global considerations or constraints are non-existent. It will be very interesting to see how far the operator analysis of geometric transition can be applied: cases of compact Calabi-Yau manifolds and also Fano varieties will be the most interesting examples to study.

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Appendix: Hopf link invariants and Topological vertex

The Hopf link invariants $W_{R_1,R_2}(q)$ of $SU(N)$ Chern-Simons gauge theory at level k depends on the parameter $q = \exp(\frac{2\pi i}{N+k})$. It is related to the string coupling g_s via $q = e^{ig_s}$. The string coupling g_s is the parameter of genus expansion of topological string amplitudes. Let μ_j be the number of boxes in the j-th row of the Young diagram R. We define two integers ℓ_R and κ_R by

$$\ell_R = \sum_{j=1}^{d(R)} \mu_j \ , \quad \kappa_R = \ell_R + \sum_{j=1}^{d(R)} \mu_j(\mu_j - 2j) \ , \tag{A.1}$$

where $d(R)$ is the number of rows of the diagram R. We note that the set of integers $\{\mu_j\}_{j=1}^{d(R)}$ defines a partition of ℓ_R. According to [5] the topological vertex is given by

$$C_{R_1,R_2,R_3} = q^{\frac{\kappa_{R_2}}{2}+\frac{\kappa_{R_3}}{2}} \sum_{Q_1,Q_2} N_{Q_1 Q_2}^{R_1 R_3^t} \frac{W_{R_2^t Q_1} W_{R_2 Q_2}}{W_{R_2}} \ , \tag{A.2}$$

where

$$N_{Q_1 Q_2}^{R_1 R_3^t} = \sum_{Q} N_{Q Q_1}^{R_1} N_{Q Q_2}^{R_3^t} \ , \tag{A.3}$$

and the tensor $N_{R_1 R_2}^{R_3}$ is the branching coefficient, or the multiplicity of the representation R_3 contained in the tensor product representation $R_1 \otimes R_2$.

In [20] a formula of the Hopf link invariants W_{R_1,R_2} has been obtained. When written in terms of Schur functions it is given by

$$W_{R_1 R_2}(q) = s_{R_1}(q^{\rho}) s_{R_2}(q^{\mu^{R_1}+\rho}) \ . \tag{A.4}$$

where $q^{\mu+\rho}$ (q^{ρ}) means that we make the following substitution;

$$s_R(x_i = q^{\mu_i - i + \frac{1}{2}}) \quad (s_R(x_i = q^{-i+\frac{1}{2}})) \ . \tag{A.5}$$

Quantum dimension of the representation R is defined by

$$dim_q R \equiv W_R(q) = W_{R\bullet}(q) = s_R(q^\rho). \tag{A.6}$$

An alternate definition of the quantum dimension is

$$dim_q R = \frac{q^{\kappa_R/4}}{\prod_{(i,j)\in R}[h(i,j)]} \tag{A.7}$$

where $h(i,j)$ denotes the hook length defined by

$$h(i,j) = \mu_i - i + \mu_j^t - j \ . \tag{A.8}$$

By looking at the formula (A.7) of $W_R(q)$, we see the following relation for the Schur polynomials at the specialized value;

$$s_{R^t}(q) = q^{-\kappa_R/2} s_R(q) = (-1)^{\ell_R} s_R(q^{-1}) \ . \tag{A.9}$$

Recall the definition of the skew Schur function

$$s_{R_1/R}(x) = \sum_{R_2} N_{RR_2}^{R_1} s_{R_2}(x) \ . \tag{A.10}$$

Hence we have

$$C_{R_1,R_2,R_3} = q^{\frac{\kappa_{R_2}}{2}+\frac{\kappa_{R_3}}{2}} s_{R_2^t}(q^\rho) \sum_{Q_3} s_{R_1/Q_3}(q^{\mu^{R_2^t}+\rho}) s_{R_3^t/Q_3}(q^{\mu^{R_2}+\rho}) \ . \tag{A.11}$$

The formula of topological vertex in terms of the special values of the skew Schur functions was first given in [21]. The above expression is slightly different from [21], but this is more convenient for our purpose. By taking $R_2 = \bullet$ in (A.11) and using the cyclic symmetry of C_{R_1,R_2,R_3}, we obtain

$$W_{R_1R_2} = q^{\frac{\kappa_{R_2}}{2}} C_{\bullet R_1 R_2^t} = q^{\frac{\kappa_{R_1}}{2}+\frac{\kappa_{R_2}}{2}} \sum_Q s_{R_2^t/Q}(q^\rho) s_{R_1^t/Q}(q^\rho) \ . \tag{A.12}$$

Thus we have obtained a manifestly symmetric form of $W_{R_1R_2}$. On the other hand, if we take $R_3 = \bullet$ in (A.11), only the trivial representation contributes in the summation over Q and

$$W_{R_1R_2} = q^{\frac{\kappa_{R_2}}{2}} C_{\bullet R_1 R_2^t} = q^{\frac{\kappa_{R_2}}{2}} s_{R_2^t}(q^\rho) s_{R_1}(q^{\mu^{R_2}+\rho}) = s_{R_2}(q^\rho) s_{R_1}(q^{\mu^{R_2}+\rho}) \ . \tag{A.13}$$

After the exchange of R_1 and R_2 we find our original expression of $W_{R_1R_2}$.

Once the topological vertices are expressed in terms of the (skew) Schur functions, a summation over representations may be performed by using the following formulas;

$$\sum_R s_{R/R_1}(x)s_{R/R_2}(y) = \prod_{i,j\geq 1}(1-x_iy_j)^{-1}\sum_Q s_{R_2/Q}(x)s_{R_1/Q}(y) \ , \tag{A.14}$$

$$\sum_R s_{R/R_1}(x)s_{R^t/R_2}(y) = \prod_{i,j\geq 1}(1+x_iy_j)\sum_Q s_{R_2^t/Q}(x)s_{R_1^t/Q^t}(y) \ , \tag{A.15}$$

For example the formula assumed in [9] and proved as Proposition 1 in [7] can be derived as follows;

$$\begin{aligned} K_{R_1R_2}(Q) &:= \sum_{R'} Q^{\ell_{R'}} W_{R_1R'}(q)W_{R'R_2}(q) \\ &= W_{R_1}(q)W_{R_2}(q)\sum_{R'} s_{R'}(Qq^{\mu^{R_1}+\rho})s_{R'}(q^{\mu^{R_2}+\rho}) \\ &= W_{R_1}(q)W_{R_2}(q)\prod_{i,j\geq 1}\left(1-Qq^{h_{R_1R_2}(i,j)}\right)^{-1} \ , \end{aligned} \tag{A.16}$$

where we have defined the "relative" hook length $h_{R_1R_2}(i,j)$ by

$$h_{R_1R_2}(i,j) := \mu_i^{R_1} - i + \mu_j^{R_2} - j + 1 \ . \tag{A.17}$$

When $R_1 = R, R_2 = R^t$ it reduces to the standard hook length.

Let us introduce the following functions

$$f_R(q) = \frac{q}{(q-1)}\sum_{i\geq 1}(q^{\mu_i - i} - q^{-i}) \ , \tag{A.18}$$

$$\widetilde{f}_{R_1R_2}(q) = \frac{(q-1)^2}{q} f_{R_1}(q)f_{R_2}(q) + f_{R_1}(q) + f_{R_2}(q) + \frac{q}{(q-1)^2} \ . \tag{A.19}$$

We have then the following lemma [7];

Lemma

$$\prod_{i,j\geq 1}\left(1-Qq^{h_{R_1R_2}(i,j)}\right) = \prod_{k=1}^{\infty}\left(1-Qq^k\right)^k \prod_k \left(1-Qq^k\right)^{C_k(R_1,R_2)} \ ,$$

where $C_k(R_1,R_2)$ are the expansion coefficients of $\widetilde{f}_{R_1R_2}(q)$

$$\widetilde{f}_{R_1R_2}(q) = \sum_k C_k(R_1,R_2)q^k + \frac{q}{(q-1)^2} \ . \tag{A.20}$$

By this lemma we find that

$$\prod_{i,j\geq 1}\left(1-Qq^{h_{R_1R_2}(i,j)}\right)^{-1}=\exp\left(\sum_{n=1}^{\infty}\frac{Q^n}{n}\frac{q^n}{(q^n-1)^2}\right)\cdot\prod_k\left(1-Qq^k\right)^{-C_k(R_1,R_2)}\ , \quad \text{(A.21)}$$

which is Proposition 1 in [7].

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